A New Combination Rule Based on the Average Belief Function

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***Abstract*—An important tool in modelling and reasoning under uncertainty is the Dempster Shafer theory (DST) of evidence. Its widespread application can be attributed to its capability to handle uncertainty due to randomness and non-specificity. The Dempster Shafer (DS) rule of combination offers opportunity to fuse pieces of evidence from different independent sources. However, the DS rule of combination is often prone to counter-intuitive results when pieces of evidence are highly conflicting. To overcome this issue, several methods have been proposed. In this work, we propose a new rule of combination based on the average belief function. The performance of the proposed method was compared with some existing approaches using numerical examples.**
Keywords -Uncertainty, Dempster Shafer rule of combination, belief function.

I. Introduction

A fundamental issue in reasoning and knowledge representation under uncertainty is how to fuse pieces of evidence in an appropriate manner. The Dempster Shafer rule of combination of evidence has been widely deployed to combine pieces of evidence from different independent sources. This tool suffers a major limitation due to its susceptibility to counter- intuitive results when the pieces of evidence are highly conflicting [2], [1]. An attempt to address this problem has given birth to two schools of thought among researchers. One school of thought believe in the modification of the traditional Dempster Shafer rule of combination. This has led to a few alternative combination rules. In [3], a modified approach was proposed where the conflicting mass is assigned to the unknown state. Similarly, in [4], an alternative method was proposed, in his case, the conflicting mass is assigned to the empty set. A disjunctive consensus rule was proposed in [5]. The other school of thought believes in the modification of basic probability of assignment (bpa). To overcome this, they do the preprocessing of evidence prior to the application of Dempster combination rule. A simple average combination rule was proposed in [6]. In [7], a weighted combination of evidence was proposed. All these ideas have produced good results.

In this work, we propose a new combination rule based on error measure between every belief function and the average belief function. In this approach, weights are assigned to each body of evidence based on the value of error between their belief functions and the average belief function of all pieces of evidence. The smaller the deviation, the higher the weight of the associated piece of evidence. The obtained weight is then used to compute the weighted average of evidence before applying the traditional Dempster-Shafer rule of combination. The rest of the paper is organized as follows: Section II dwells on the basics of Dempster- Shafer theory of evidence, the focus of section III is the new combination rule. Numerical examples are presented in section IV to illustrate the performance of the proposed method. The focus of Section V is the presentation of simulation results and discussion. Finally, in section VI, the conclusion is presented.

II. Dempster- Shafer Theory (DST)

DST is a mathematical theory of evidence proposed in [8], which is an extension of the work in [9]. It is a generalization of probability theory. In evidence theory, probabilities are assigned to subsets instead of mutually exclusive singletons. Evidence theory can handle uncertainty better than probability theory [3]. Evidence is characterized by functions which include: basic probability assignment (BPA), belief function, plausibility function, and commonality function[10].

A. Basic Functions

Let $\Omega = \{\theta_1, \theta_2, ..., \theta_N\}$ be the frame of discernment, a set of mutually exhaustive and exclusive hypothesis. A power set 2^Ω is the set of all possible subsets of Ω.
For all $A \subseteq \Omega$, the mass function also known as basic probability of assignment $m : 2^\Omega \rightarrow [0, 1]$ satisfies the following conditions:

$$\sum_{A \subseteq \Omega} m(A) = 1 \tag{1}$$

$$m(\emptyset) = 0 \tag{2}$$

The belief function $Bel : 2^\Omega \rightarrow [0, 1]$ is defined as

$$Bel(A) = \sum_{B \subseteq A} m(B) \tag{3}$$

The plausibility function $Pl : 2^\Omega \rightarrow [0, 1]$ is defined as

$$Pl(A) = \sum_{A \cap B \neq \emptyset} m(B) \tag{4}$$

The commonality function $Q : 2^\Omega \rightarrow [0, 1]$ is defined as

$$Q(A) = \sum_{B \supseteq A} m(B) \tag{5}$$

B. DS Rule of Combination

The DS combination rule was introduced in [8]. It provides a mathematical framework to obtain combined evidence when multiple independent pieces of evidence are available. Let m_1 and m_2 be mass functions produced by two independent pieces of evidence defined on the same frame of discernment Ω. The combined mass is defined as [8]

$$m(A) = \frac{\sum_{X\cap Y=A} m_1(X)m_2(Y)}{1-\sum_{X\cap Y=\emptyset} m_1(X)m_2(Y)} \tag{6}$$

$\forall A, X, Y \subseteq \Omega$ and $A \neq \emptyset$.

C. Evidence Distance

Evidence distance is used to measure lack of similarity between two pieces of evidence. In [11], a distance function was presented to measure distance among the basic probability assignments (bpa). The distance between two mass functions m_1 and m_2 denoted by $d(m_1, m_2)$ can be defined as

$$d(m_1, m_2) = \sqrt{\frac{1}{2}(\overrightarrow{m_1} - \overrightarrow{m_2})^T D(\overrightarrow{m_1} - \overrightarrow{m_2})} \tag{7}$$

$$D(A, B) = \frac{|A\cap B|}{|A\cup B|} \quad A, B \subseteq \Omega \tag{8}$$

Where $\overrightarrow{m_1}$ and $\overrightarrow{m_2}$ are vector representations of m_1 and m_2 respectively.$D(A,B)$ is a 2^N by 2^N matrix. The measure of error for belief function approximation adopted in [12], is a kind of distance function which is defined as [13] .

$$d(m_1, m_2) = \sum_{A\subseteq\Omega} |Bel_1(A) - Bel_2(A)| \tag{9}$$

III. THE NEW COMBINATION RULE

The proposed method is based on the error measure between a given belief function and the arithmetic mean of all belief functions. A body of evidence whose belief function is closer to the average belief function is assigned a higher weight than the one that is farther away from the average belief function. Consider an m by n matrix μ, with m propositions and n pieces of evidence. Such that each column represents the basic probability assignment from each evidence.

$$\mu = \begin{bmatrix} \mu_{11} & \dots & \mu_{1j} & \dots & \mu_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mu_{i1} & \dots & \mu_{ij} & \dots & \mu_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mu_{m1} & \dots & \mu_{mj} & \dots & \mu_{mn} \end{bmatrix} \tag{10}$$

The proposed methods entails the following steps

- Due to one to one mapping among mass, belief, and plausibility functions [8], we can use the belief function to determine the weight of each piece of evidence. Therefore, the first step is to transform the basic probability assignment of each evidence into belief function.
- Construct a m by n matrix p, with m propositions and n pieces of evidence, such that each column represents belief function for each evidence. i.e. p_{ij} is the belief given to proposition i by evidence j.

$$p = \begin{bmatrix} p_{11} & \dots & p_{1j} & \dots & p_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{i1} & \dots & p_{ij} & \dots & p_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{m1} & \dots & p_{mj} & \dots & p_{mn} \end{bmatrix} \tag{11}$$

- Compute the average belief of evidence, such that the average belief to each proposition is defined as

$$\bar{p_i} = \frac{1}{n}\sum_{j=1}^{n} p_{ij}, i = 1, 2, ..., m, and j = 1, 2, ..., n \tag{12}$$

- Adopting a distance measure similar to the one defined in [12], for every belief function, compute its deviation from the average belief function, which is denoted by α_j and defined as

$$\alpha_j = \sum_{i=1}^{m} |\bar{p_i} - p_{ij}| \tag{13}$$

- Calculate the weight β_j of every belief function and it is defined as.

$$\beta_j = exp(-\alpha_j) \tag{14}$$

- The normalized weight becomes

$$w_j = \frac{\beta_j}{\sum_{l=1}^{n} \beta_l}, l, j = 1, 2, ..., n \tag{15}$$

Thus the weight vector becomes

$$w = [w_1, w_2, ..., w_n]^T \tag{16}$$

- Compute the Weighted Average of Evidence m_{wae} as

$$m_{wae} = \mu w \tag{17}$$

- Apply DS combination rule to $m_{wae}(n-1)$ times [6].

IV. NUMERICAL EXAMPLES

In this section, we present two numerical examples to demonstrate the performance of the proposed method. The performance of the new combination rule is then compared with some existing methods.

A. Example I

Suppose the actual target being observed by a multi-sensor target recognition system comprising of five sensors is target A. Let us assume that the target belongs to one of the three classes A, B, and C. The information reported by the five different sensors are shown in Table I

B. Example II

This example is taken from [14]. The frame of discernment is $\Omega = \{A = Fighter, B = Bomber, C = Commercial\}$. The basic probability assignment from the five pieces of evidence is shown in Table II.

Table I
NUMERICAL EXAMPLE I

Proposition	m_1	m_2	m_3	m_4	m_5
m(A)	0.41	0.00	0.58	0.55	0.60
m(B)	0.29	0.90	0.07	0.10	0.10
m(C)	0.30	0.10	0.00	0.00	0.00
m(AC)	0.00	0.00	0.35	0.35	0.30

Table II
NUMERICAL EXAMPLE II

Proposition	m_1	m_2	m_3	m_4	m_5
m(A)	0.50	0.00	0.55	0.55	0.60
m(B)	0.20	0.90	0.10	0.10	0.10
m(C)	0.30	0.10	0.35	0.35	0.30

V. SIMULATION RESULTS AND DISCUSSION

A. *Simulation Results*

After applying $(10) - (15)$, the normalized weight for example I are $w_1 = 0.3355, w_2 = 0.0458, w_3 = 0.1901, w_4 = 0.2143$ and $w_5 = 0.2143$. The weights are then used to modify the evidence by using (17) to compute the weighted average of evidence. The results of the weighted average of evidence is given as $m_{wae}(A) = 0.4943, m_{wae}(B) = 0.1946, m_{wae}(C) = 0.1052, m_{wae}(AC) = 0.2059$. With the application of the traditional Dempster shafer rule of combination 4 times, the final fused masses are $m(A) = 0.9813, m(B) = 0.0016, m(C) = 0.0149, m(AC) = 0.0022$.

Similarly, for example II, we obtained the following normalized weights: $w_1 = 0.3199, w_2 = 0.0369, w_3 = 0.2144, w_4 = 0.2144$ and $w_5 = 0.2144$. The corresponding weighted average evidence is given as $m_{wae}(A) = 0.5244, m_{wae}(B) = 0.1615, m_{wae}(C) = 0.3141$. The final fused masses are $m(A) = 0.9261, m(B) = 0.0026, m(C) = 0.0713$.

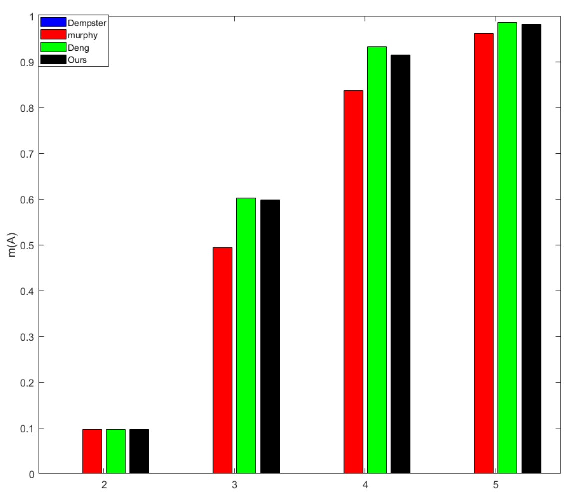

Figure 1. The comparison of combined BPA for proposition A for Example I.

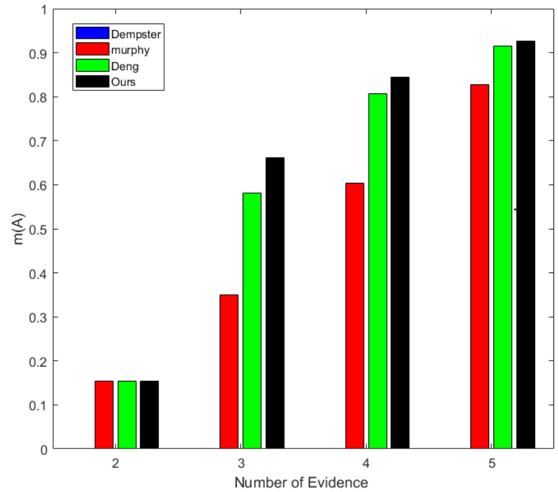

Figure 2. The comparison of combined BPA for proposition A for Example II.

B. *Discussion*

It is very noticeable, that evidence m_2 is highly conflicting with the remaining sources of evidence and therefore it is considered as highly unreliable. This is because its weight is just 0.0458 and 0.0369 for example I and II respectively. The proposed method was compared with the Dempster-Shafer rule, Murphy simple average and Deng weighted average method. Our proposed method competes favorably with Deng approach and produce more reasonable results than the Dempster's rule and Murphy's approach. The results of the two examples are shown in Table III and IV respectively. The comparison of the combined mass by different combination rules for proposition A is shown in Figure 1 and Figure 2 for Example I and II respectively.

The results for proposition A using Dempster Shafer's rule is represented by the blue legend, since the final fused mass is zero, it cannot be observed from the bar plot as shown in Figure 1 and Figure 2. The inability of the Dempster's combination rule to produce reasonable results when faced with conflicting pieces of evidence was obvious. The results generated based on Dempster's rule were illogical and counterintuitive as portrayed by the two examples. The combined mass for proposition A using Dempster's rule remains at zero regardless of updating the fused mass with the remaining pieces of evidence. In example I, our proposed method performs better than Murphy s approach, however, the performance of Dengs method is slightly better than ours. In case of Example II, our proposed method is better than all other methods. The performance of Deng's method is very close to ours. The fused masses produced by Murphy, Deng and our approaches agree with the intuition.

VI. CONCLUSION

In this work, a new combination rule based on the error measure between the average belief function and every belief

Table III
FUSED RESULTS FOR NUMERICAL EXAMPLE I

Methods		$m_1 - m_2$	$m_1 - m_3$	$m_1 - m_4$	$m_1 - m_5$
Dempster [9]	A	0.0000	0.0000	0.0000	0.0000
	B	0.8969	0.6350	0.3321	0.1422
	C	0.1031	0.3650	0.6679	0.8578
Murphy [6]	A	0.0964	0.4938	0.8362	0.9620
	B	0.8119	0.4180	0.1147	0.0210
	C	0.0917	0.0792	0.0410	0.0138
	AC	0.0000	0.0082	0.0081	0.0032
Deng [7]	A	0.0964	0.6021	0.9330	0.9851
	B	0.8119	0.2907	0.0225	0.0017
	C	0.0917	0.0991	0.0354	0.0096
	AC	0.0000	0.0082	0.0092	0.0039
Ours	A	0.0964	0.5984	0.9141	0.9813
	B	0.8119	0.2632	0.0256	0.0016
	C	0.0917	0.1358	0.0560	0.0149
	AC	0.0000	0.0027	0.0043	0.0022

Table IV
FUSION RESULTS FOR NUMERICAL EXAMPLE II

Methods		$m_1 - m_2$	$m_1 - m_3$	$m_1 - m_4$	$m_1 - m_5$
Dempster [9]	A	0.0000	0.0000	0.0000	0.0000
	B	0.8571	0.6316	0.3288	0.1404
	C	0.1429	0.3684	0.6712	0.8596
Murphy [6]	A	0.1543	0.3500	0.6027	0.8273
	B	0.7469	0.5224	0.2627	0.0863
	C	0.0988	0.0792	0.1346	0.0863
Deng [7]	A	0.1543	0.5816	0.8060	0.9149
	B	0.7469	0.2439	0.0482	0.0082
	C	0.0988	0.1745	0.1458	0.0769
Ours	A	0.1543	0.6612	0.8433	0.9261
	B	0.7469	0.1572	0.0196	0.0026
	C	0.0988	0.1817	0.1371	0.0713

function was proposed. The proposed method assigns weight to individual pieces of evidence based on the measure of deviation between their belief function and the average belief function. The closer the belief function of an evidence to the average belief function of all pieces of evidence, the higher the weight and vice versa. The proposed method was able to overcome the counter-intuitive issue associated with the traditional Dempster Shafer rule of combination. Two numerical examples were used to verify the rationality of the proposed method, and its performance was comparable with some of the existing alternative methods.

ACKNOWLEDGMENT

This work is supported by the Air Force Research Laboratory and OSD under agreement number FA8750-15-2-0116. The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of Air Force Research Laboratory and OSD or the U.S. Government.

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